# Spaces beyond the Space

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We live in three-dimensional space, called by mathematicians Euclidean space, in honor of Euclid, a geometer from Alexandria who attempted more than 2000 years ago to express our intuitive spatial perception in a list of axioms. Starting from the last century, especially after the work by Bernhard Riemann, the idea of space kept evolving and moving further and further away from its Euclidean origin. This was motivated by the needs of pure mathematics, as well by the development of the sciences, first physics, joined later by chemistry, biology, information theory, etc.

Most spaces we encounter in mathematics and science cannot be visualized within our three-space: these are not like curves or surfaces; they do not come to us in visual images. Even if such spaces are intimately related to Euclidean space and we depend on them every moment in life, we do not consciously perceive them, yet exploit their properties without ever noticing them, e.g., in the course of locomotion. (The last thing a fish is aware of is water.)

How can one see such spaces; how can one study them; how can one communicate acquired knowledge to a machine—a robot designed, for example, to move in space with the agility of an animal or a human?

What aids us and what makes geometry as we practice it nowadays possible is an amazing plasticity of our spacial intuition, built into our visual and motor systems. Studying a particular branch of geometry is similar to learning to ride a bicycle: it seems impossible at first, but once you have learned, you cannot tell what the difficulty was. In geometry, however, we deal with a tower of bicycles, one idea on the top of another.

In this lecture, I will try to take the audience on an easy (maybe not that easy) ride. My purpose is twofold: firstly I want to demonstrate how unexpectedly rich and intricate the structure of the ordinary three-space is. Secondly I want to give you a glimpse of the world of invisible spaces surrounding us. Imperceivable with the naked eye, these can be seen along with the three-space when we translate our spacial intuition into a suitable geometric language.

In order to understand our three-space we have to learn how to ask questions about it. Start with a few of them. What are the structure and the essential properties of the space around us? By what means do we see and perceive it? How do we manage to move in space? What is so special about the space that makes motion possible at all?

Imagine a prehistoric geometer putting these questions to his friend, a smart Cro-Magnon hunter. Never mind that a genius able to arrive at such questions without the back-up of civilization and centuries of scientific thought has never been recorded in human history; imagine nevertheless. If the hunter were a comparable super-genius he would start thinking and trying to comprehend the meaning of the questions. Then he would realize that the questions made sense but he didn't know the answers. If he were just an ordinary genius, he could reply that the questions were apparently meaningless and unquestionably useless. Space is just space: birds, fish, bugs—all animals know what the space is. They see it perfectly, as is proved by their ability to move so efficiently. Why bother asking silly questions when there are truly profound ones: why does the universe exist? What is the meaning of life?

The difficulty in grasping the essentials of ordinary space is somewhat paradoxical: we know it too well. Our visual and motor systems have evolved, molded by the space we inhabit, to an unbelievable level of performance. (That took about half a billion years since the Cambrian explosion.) Just think about it: a messy flow of reflected light waves chemically excites the retina in our eyes. By a neurological process we have hardly started to unravel, this creates consistent spacial images in the mind of an observer. These images, besides being consistent, are faithful to the true geometry of the space. Convincing evidence is provided, for example, by the perfect coupling between the vision and the motor systems of higher animals that allows locomotion as well as other kinds of purposeful movements in space compatible with the laws of geometry and mechanics. (Light is not indispensable: bats and some dolphins, for instance, "see" exclusively with their ears.)

Our unconscious mind harbors a treasure of knowledge about space, a wealth of geometric (as well as mechanical) intuition but we have no direct access to this treasure since it resides far from the logical and linguistic parts of our brain that are responsible for sequential reasoning. With a painful effort through the centuries, by asking and answering correct (and often apparently meaningless) questions, we have arrived at the present understanding of our habitual space as well as other spaces; the acquired knowledge, unlike the underlying intuition, can be expressed and communicated in words and thus incorporated into culture.

What kind of language is capable of capturing the essence of space and turning it into words? What can guide us in making up such a language?

There are several interlinked languages corresponding to different branches of geometry, where each branch is concerned with a particular class of objects. Distance (also called metric) geometry is concerned with spaces equipped with distances between points. This is the most intuitive branch of modern geometry as I shall explain in detail later in this lecture.

The most active these days is symplectic geometry, where one studies a generalized notion of area assigned to high dimensional spaces, such as the phase spaces of physical systems that obey Hamiltonian mechanics.

Algebraic equations and their solutions are studied nowadays in the context of algebraic geometry, where the simplest objects are lines, circles and ellipses in the plane as well as spheres and ellipsoids in the three-space. A particular branch of algebraic geometry, called Diophantine geometry, is concerned with solving equations where the required solution must be given by integers.

You may justifiably feel disconcerted with all these strange words and obscure allusions. I wish I could tell you: “do not worry, these are just words, the truth behind them is quite simple.” The first is correct, the terminology is rather arbitrary. Nobody knows, for example, how the word “symplectic” crept in here. But the truth behind the words is incomparably more profound, and, alas, there is no King’s road to mathematics. Even if you are a professional geometer, who knows everything about the distance and algebraic equations, it will take several lectures to attend and hours of concentrated thinking to grasp the meaning of “symplectic,” while actual learning of the subject takes years. (I know, by personal experience, how long it takes to learn and appreciate a new mathematical theme and how difficult it is to communicate ideas to a most sympathetic professional audience. My first lecture on symplectic geometry in Tokyo about 17 years ago was a complete disaster.)

This is not so disappointing as it looks: as a matter of comparison, take music. The title of a particular piece may look strange, but it has little to do with what it stands for. Learning music, in the way one learns mathematics, is reading the scores till the music sounds in your head and you become able to reproduce it on a musical instrument. Once you have learned it (I speak of mathematics), you will almost never forget it: the symphony of ideas is always with you. What is missing in mathematics is something corresponding to public performance of music, transforming the musical idea of a composer into sounds and bringing it to everybody.

In the beginning, geometry was rather simple and close to naive intuition; at least it looks this way from the heights of 21st century. The first systematic language was laid down by Euclid in about 300 B.C. This language was designed for describing one particular space, the space we live in. Following the mathematical convention we call this space, or rather its mathematical description suggested by Euclid, the "Euclidean three-space."

Speaking about geometric languages, I do not want to project an impression that geometry is solely concerned with descriptive languages. It is no more true than thinking that poetry is all about poetic language or music is about scores. There is a continuous feedback between the intuitive geometry coming in vague images and the abstract geometry depending on a precise language. Amazingly, many intuitive geometric ideas can be cast into a formally consistent language, making an intricate web of mutually intertwined statements. On the other hand, many intuitively appealing ideas (not only in geometry) turned out to be wrong, if not plain meaningless. Some still wait to be either formalized or discarded.

An example of the most successful idea is that of symmetry, which came from geometry and then penetrated all branches of science. Everybody perceives the perfect symmetry of the circle in the plane and that of the round two-sphere in the three-space. This symmetry is so perfect that it is hard to put your finger on it. In fact, it reflects an even more fundamental symmetry of the Euclidean space itself as I shall explain later.

It is easier to appreciate less uniform symmetry such as, for example, that of "Platonian solids (bodies)," the five convex polyhedra[Fig. 1].

(a) Three-simplex (regular triangular pyramid): it has four vertices and four regular triangular faces.

(b) Cube: eight vertices and six square faces.

(c) Octahedron: six vertices and eight regular triangular faces.

(d) Dodecahedron: this is less obvious, but it has twenty vertices and twelve regular pentagonal faces.

(e) Icosahedron: twelve vertices and twenty regular triangular faces.

There are no other polyhedra with such high symmetry. This is intuitively clear: everything you try to make does not look so perfect. Yet, it is not obvious at all how to pinpoint the essential property of the great five that distinguishes them from their less symmetric brethren. The proper language, nowadays called "group theory," adequately capturing the idea of symmetry, was developed only in the 19th century, more than the two thousand years after Plato. The essential symmetry property of these bodies can be described as follows.

Take a face of such a polyhedron, say $F$ (this may be a regular triangle, a square, or a regular pentagon depending on which of the five polyhedra we deal with), and a vertex $V$ in $F$. Now repeat the choice all over again: take another face $F'$ (possibly equal to the first $F$) and a vertex $V'$ in $F.'$ Then, no matter how we have picked these, there exists a rotation of our body around the center that moves $F \to F'$ and $V \to V.'$

Notice that this rotation moves the whole body exactly into itself so that each vertex (not only the chosen $V$) goes to a vertex, edges go to edges and faces to faces. (There are many rotations that transport *V to* $V'$ without matching the rotated copy of the body with the unmoved one. Yet, if a face goes to a face, than all of the polyhedron must go into itself due to its symmetry.)

This is clear by looking at the picture. What is less clear, and this is a non-trivial (but not difficult) mathematical theorem, is that every convex polyhedral body with the above property (i.e., where there are rotations moving faces, edges, and vertices around with the same freedom) is necessarily one of the five. Apparently, this theorem was on the mind of Plato who distinguished his solids as the most perfect ones.

The five Platonian solids are followed by thirteen "Archimedean solids" [Fig. 2]. These are also highly symmetric but not so much so as the Platonian ones. Apparently, Archimedes, as well as Plato, was guided by his intuitive idea of symmetry, similar to our theory of groups, but the later geometers have lost this intuition. I came to this conclusion in the following way. There was a definition of Archimedes' bodies made somewhere in the 18th or 19th century similar to the naive definition of a regular polygon in the plain, namely, a polygon in the plane is called regular if all its edges have equal lengths and the angles between all adjacent faces are mutually equal.

This definition gives (the proof is simple) exactly those $n$-gons that we intuitively perceive as regular, and where precise symmetry property distinguishes them is the same as for polyhedra, even easier: there is a rotation of such a polygon moving one vertex to another arbitrarily chosen one.

There is an extra symmetry to these polygons: they are mirror-symmetric with respect to the lines normal to the edges at their centers and the Platonian bodies possess mirror symmetries as well. The mirror symmetry with respect to a line in the plane can be implemented by rotating the plane 180 degrees in space around this line but the mirror symmetry in space is harder to grasp: one's left hand is (almost) equal to the mirror image of the right one, but the two hands cannot be matched by moving in three-space.

The above edge-angle definition for regular polygons captures the Platonian bodies as well: Platonian bodies are exactly those polyhedra where all faces are mutually equal regular polygons and where the angles between all adjacent faces are equal to each other (actually, the latter follows from the former, but this needs a non-trivial argument).

A similar statement about the Archimedes' bodies was cited and "proved" in all advanced text books on geometry starting from the 19th century. In the late 1950s, a Russian geometer Ashkenazi found a counterexample that was acclaimed as a "perfect body missed by Archimedes" [Fig. 3]. When I was a student, a Leningrad geometer, Victor Zalgaller, suggested to me to look at this newly discovered body and derive from it other geometric objects, namely, symmetric star-shaped polyhedra that were known to be associated with the original 13 convex polyhedra of Archimedes. I failed, no nice star-shaped has emerged from the new polyhedron. Eventually I realized that the new body, although satisfying the classical definition, was not as symmetric as the Arcadian ones. So (to make myself feel better at my failure) I decided that it wasn't Archimedes but the author(s) of the accepted definition of Archimedes' bodies who had missed the point. (I guess this definition had been invented before group theory became a common working tool in geometry; still there is a confusion in the popular literature.)

The branch of geometry I will present to you, called distance geometry and also metric geometry, is concerned with distances between points. In the Euclidean three-space, the ordinary, also called Euclidean, distance is measured by the length of straight segments between points. To make this non-ambiguous, one needs to fix a scale, that is, to declare some segment a unit, say one meter or one centimeter, and then the distance becomes a pure number assigned to every pair of points in space.

It would not be worth emphasizing and assigning a name to this branch of geometry were it not for two reasons. First, all essential properties of the Euclidean space can be expressed in the language of distance. Second, there are many spaces, some of them visible inside our (Euclidean) three-space that come along with different kind of metrics, i.e., with various prescriptions of distances between their points.

Let us look at the surface of the earth. A straight spacial segment between Kyoto and Paris, for example, passes a few thousand kilometers deep underground. The Euclidean distance is no good for the travelers who stay on the surface of the Earth.

A more practical distance is the length of the shortest path between two point locations on the *surface* of the Earth. This distance is significantly greater than the Euclidean one, since the Earth is not flat. One may consider even larger distances, where one only allows routes between two points that are available to given means of transportation. For far-lying points, for instance, the (nearly) shortest path would be achieved by an air routes if the flights were not a subject to non-geometric constraints, from one location to another. Instead of time one could use the minimal amount of fuel or the cheapest price for travel as a measure of distance. Some of these conform to the accepted idea of distance, but the resulting geometries are rather arbitrary and, although practically useful, are not especially attractive to aesthetically minded mathematicians.

The idea of minimal path is used for introduction of metrics (i.e., prescribed distances) on surfaces in the Euclidean space: the metric, sometimes called "intrinsic metric," is defined by declaring the distance between two points to be equal to the length of the shortest path on this surface joining the points. (This distance between two points on the surface is greater than the Euclidean distance unless the surface contains the straight segment joining these points in space.)

For example, if we take a round sphere (a good approximation to the Earth's, surface on the large scale), then shortest are segments of great circles on the sphere: these circles are seen by cutting sphere with planes passing through the center of the sphere. ("Sphere" in mathematical jargon refers to the two-dimensional surface that bounds a round ball.)

The above is childishly simple, yet it requires an effort to keep in mind two images: the Earth, the huge mass of land and water, and that of a spherical surface, a soup bubble fixed in space or the surface of a globe.

In order to understand the most essential property of the space, expressible in the language of distance, we first try the two-plane, which is somewhat easier to grasp. Think of the plane as the top of a perfectly flat table and imagine it is covered by a sheet of transparent paper. Then, as we know from experience, one can freely slide the paper on the table without stretching or wrinkling it. In fact we have the complete freedom of movement: if one marks one point on the paper and another one at a different location on the table, one can move the paper by sliding it on the table so that the two marks will match; if one has two marked points on the paper and two on the table, one can match the two pairs by sliding the paper over the table, provided the distance between the points on the paper equals the distance between the points on the table.

The same remains true for many marked points. For example, let us mark five points, on the paper, call them p1, *p*2, *p*3, *p*4, *p*5, and also five on the table, named *P*1, *P*2, *P*3, *P*4, *P*5. Then we can slide the paper such that, simultaneously, each of the *p*-points on the paper will go to the *P*-point on the table with the same number (i.e. *p*1→*P*1, *p*2→*P*2, *p*3→*P*3, etc.), provided the mutual distances between the five *p*′s equal the corresponding distances for *P*′s: the distance between *p*1 and *p*2 must equal that between *P*1 and *P*2, etc., for all pairs of *p*-points.

There are ten such pairs altogether, and thus the possibility of the matching slide needs the equality of two arrays of ten numbers, that is, ten equalities between the corresponding numbers expressing the distances. (To be true to mathematical rigor, this is not correct as it stands: if one marks five points on the paper with the fingertips of the left hand and another five on the table with the right hand, then, in order to match the left and the right markings, one needs to combine sliding the paper over the table with a reflection of the plane in a line, that is, a 180 degree rotation of the sheet of paper in space around a line.)

To appreciate the significance of this one needs to look at other surfaces. Imagine we have a curved table also covered by an accordingly curved transparent paper perfectly matching the surface of the table. If the top of the "table" is spherical, a globe serving for a table, then all we have said about the plane remains true: we can slide the spherical paper on the sphere as much as we want.

Less obviously, this is also true for a cylindrical (and even conical) "table": one can unroll the paper wrapping a cylinder (or a cone) into a flat sheet; then it can be slid along itself, and thus over the surface of the cylinder (cone) with the same freedom as over the flat table. However, if our table is flat at one location and has a spherical bump somewhere else, then one can not freely slide the paper anymore: the paper “remembers” the bump and it cannot be moved away from it without this bump bulging away from a flat part of the table.

Now we want to develop a similar picture for the three-space. It is hard to speak of a three-dimensional “paper” covering the space. Thus we need a better language for the case of surfaces. Instead of introducing a paper covering a surface we just create in our mind a second copy of a given surface, a truly invisible paper perfectly matching the surface. Then we can speak of a surface sliding along itself, thinking of the copy (paper) sliding over the first unmoved copy (the top of the table) without any explicit reference to such mundane objects as tables and paper.

The same language applies to any space: we may speak of moving space along itself, where the word “move” signifies the preservation of distances: if in the course of such a move, a point $p$ goes to some $P$ and another point $p'$ goes to $P'$ then the distance between $P$ and $P'$ must be equal to that between the original points $p$ and $p'$ for all possible choices of the points $p$ and $p'$.

Recall that “distance” is something attributed to a space and there may be several ways to assign distances as we have seen on the surface of the Earth. It may happen that some move is admissible for one metric, i.e., it preserves this distance, but does change another one. For example, the spherical metric on the Earth (where the shortest lines are segments of great circles in the spherical model of the Earth) is preserved by rotations of the sphere, but other metrics we described (e.g., defined by the price of tickets) are not.

The ordinary Euclidean three-space can be moved along itself with the same freedom as the plane (or the two sphere): given two configurations of points in the Euclidean three-space, say $p1$, $p2$, $p3$, $p4$, $p5$ and $P1$, $P2$, $P3$, $P4$, $P5$ with mutually equal distances between the corresponding (ten) pairs of points, then there is a motion (including symmetries in some planes) of the whole space moving one configuration into the other.

This is the basic property of our space, allowing all kinds of motions we physically experience in space. As mathematicians, we take this property as a basic axiom of the Euclidean space. Here again, to appreciate this property (axiom), one needs something to compare our space with. This is more difficult than in the case of the plane, since other three-dimensional spaces cannot be seen inside the Euclidean one. Thus we need another source of examples.

Let us return to the plane, a perfectly flat infinite table as earlier, but now instead of points let us think of straight rigid rods, e.g., thin pencils, lying on the table. Mathematically speaking, we are dealing with straight segments in the plane. We agree that the rods have unit length (say one decimeter=10 cm).

We want to consider *all* such rods on the table, or, mathematically, *all* unit segments in the plane. Of course, this is impossible to imagine: the whole table covered by pencils, where the pencils at all points are directed at all angles. It is better to think of all possible positions of a single rod. A rod can be placed anywhere on the table and moved from one position to another. So our "all" means that we allow all positions but at every moment we shall be concerned with a few of them. On the other hand, by considering all rods simultaneously we create in our mind a new space where the word "point" stands for a particular (position of a) rod on the table.

To justify this we need to identify a geometric structure in this space of rods and in the context of this lecture, "geometric structure" signifies "distance." Once we can measure the distance between rods, we can think of this rod-space as a kind of ordinary three-space with a distance different from the Euclidean one where points are just points, with no memory left of the rods.

Many mathematical constructs are obtained by "forgetting" some of the properties of the objects we study. For example, the ordinary plane is an abstraction of a flat surface in the real world, but we forget many things, e.g., the color, the texture, etc. This forgetting serves to isolate the properties we want to study. As a benefit, we find unexpected similarities between rather different objects, such as "all points in the three-space" and "all unit rods in the plane." Observe that an individual rod has little in common with a point in the space. However, the totality of rods in the plane is rather similar to the the Euclidean space insofar as we are concerned exclusively with the mutual relationship between the rods expressible in term of the distances between them.

Now we come to the crucial point: we want to find a distance between every pair of rods in the plane. There is no God-given distance. We have to invent one, but this should be done with certain prudence if we want to obtain a usable space. We follow the same idea as for the distance on a surface in space, where it is defined as the length of the shortest path between the points in the surfaces. But what is a “path” between two rods in the plane? Here there is an obvious candidate: a path is a continuous motion of the rod in the plane bringing it from one given position, say, rod 1, to another one, rod 2.

As we move a rod in some way its two ends move along in the plane. We imagine the ends of the rod are colored, say, one end is green and another blue, and thus we get two curves in the plane: one curve, the green one, goes from the first end of rod 1 to that of rod 2; and the second, the blue one, joins the second ends of the rods.

How can one measure the joint length of these two paths, the green one and the blue one together? A naive idea is to take the sum of the two. One can do that, but this is by no means the best choice. A better distance may be defined with the Pythagorean theorem in mind, which suggests taking the square root of the sum of squares of the lengths of the two curves. But the true (double) length is trickier than this. (The above is OK if the two ends move with constant, possibly different, speeds.)

Unfortunately, the correct definition, which eventually leads to a simple “natural” distance, is more complicated. The definition that is about to come is difficult to grasp, not because we want unnecessary complexity, but because Nature has not designed our brain for this kind of problem. This definition, the true definition, took the genius of Bernhard Riemann to discover following centuries of intense thinking by mathematicians and physicists. One is not supposed to absorb it easily, and the “truth” of Riemann’s definition becomes clear only a posteriori by its successful use in mathematics, science and engineering. (Definitions in mathematics are not just words assigned to objects that we already know and understand well but are acts of creation crystallizing essential, and often difficult, ideas.)

Now comes the definition. We have our rod moving somehow in the plane. Say, the full motion from one position to another takes one minute. We record the successive positions of the rod in smaller units, e.g., in seconds. Thus our motion is divided into sixty small steps: every second the rod moves from some intermediate position to the next one, which is rather close to the preceding position. At every such step, we take the two ordinary distances between the corresponding positions of the ends of the rod, one distance between two consecutive green points and the distance between the corresponding blue ones. We “add” each green distance to the corresponding blue one by the Pythagorean rule, i.e., we take the square root of the sum of their squares. Finally, we add the resulting sixty numbers and call the sum the “length of (the path of) the motion on the scale of seconds.”

If we need more precision, we can use a finer time scale, say milliseconds. Then the motion is divided in to 60,000 tiny steps, each of which is treated as earlier, and the sum of 60,000 small terms adds up to the “length of the (same) motion on the scale of milliseconds.”

This may be sufficient for most practical purposes, but mathematicians do not like (and for a good reason) arbitrarily fixed scales. Thus one takes smaller and smaller time scales (a molecular physicist would stop at femptoseconds, one millionth of the billionth part of a second), eventually defining the true, scaleless length as the limiting value of these more and more precise measurements.

Finally, we arrive at the definition of the distance between rods in the plane: imagine all possible paths between two rods, measure the above true length of every path, and take the smallest of these as the distance between the rods.

Now, why all this mess? The answer is: “It is not our choice, the nature of mathematics and physics dictates this definition.” We shall justify this below.

Granted the notion of distance, we may start asking meaningful questions. First of all, how can one determine the distance, what is the shortest path between two rods?

One may start with a guess that the shortest way to go from one rod to another is by moving the ends along straight lines. It is not so bad and gives the correct answer if the two rods are parallel in the plane. But if there is an angle between the two, e.g., the two meet at an end with an angle, such motion is impossible, since, remember, the rod is not supposed to change its length along the way.

Now, suppose the rods meet at their green ends on the plane at some angle. Then, clearly the shortest motion joining them is given by rotation of one of them around the fixed green end. Observe that there are two such rotations, clockwise and counter-clockwise. We take the one that makes an angle of less than 180 degrees. There is an unavoidable ambiguity if the blue ends point in opposite directions and both angles are exactly 180 degrees. In this case, we have two shortest paths between the rods! This makes our space quite different from the Euclidean one, where there is exactly one shortest path between given points, namely, the straight segment.

The situation here is somewhat similar to that on the sphere, where the shortest paths are segments of great circles: if two points are opposite each other on the sphere, as the North and South Poles, then there are an infinite number of shortest paths: one may travel from the South Pole to the North Pole along any meridian, they are all equally long.

Actually, one can describe the shortest paths between arbitrarily positioned rods as follows. Move the center of the rod along the straight line with constant speed and simultaneously rotate the rod around its center. There may be several such paths between two rods, and one needs the one where the overall rotation does not exceed 180 degrees.

With our complicated definition, it is not easy to show this is indeed the shortest path whose length gives us *the* distance defined via the subdivision process. One may ask: "Why could one not use the above description as *the* definition." The answer is: it is possible to do so for this particular example, but our messy definition has an advantage of being applicable to *all* similar spaces, as we shall see in a minute.

Finally, still sticking to rods in the plane, we ask a more sophisticated question: does the rod space have symmetry similar to that of the Euclidean space? For example, given two points in the rod-space, can one move the whole rod-space along itself so that one point goes to the other?

The answer is "yes" due to the following simple observation (which a mathematician would call a "trivial lemma").

Imagine we have two rods attached to the paper covering the table and then slide the paper to a new position. The rods move along with the paper and, this is the key point, the rod-distance between the attached rods doesn't change in the course of the slide. In other words, the paper slides give us motions of the rods space that do *not* change the rod-distance.

Now, we know that one can move a given rod to any other rod by such a slide; therefore, there is a bone fide motion of the rod-space moving a given point (represented by a rod in the plane) to any other point (rod). Thus the rod-space does have a kind of symmetry similar to what happens in the Euclidean space.

The above may look (to some extent justifiably) like just a silly word game, where something obvious is pompously pronounced in an obscure jargon. Never mind and turn to the next question: given two pairs of points in the rod-space with mutually equal distances between the points in the pairs, can one move the rod-space such that the first pair of points goes to the second one?

If it were the Euclidean space, the plane or the sphere, the answer would be "yes," as we have seen already. However, it is "no" in the rod-space. Thus the rod-space is less symmetric (or less movable) than the ordinary space. (It is easy to realize that no sliding of the plane can move a pair of parallel rods to non-parallel ones, although the distances may be arranged to be equal. But in order to have a convincing "no," one needs to rule out possible motions of the rod-space that do not come from any paper sidings. To do this one is obliged to enter into a more abstract reasoning that I dare not to present in this lecture.)

Tired of rods, let us turn to regular triangles of unit size positioned in three-space. Every path of motion of such a triangle is given by three curves in space, the trajectories of the three vertices. The general definition allows us to make the distance out of such triples of curves with the same ease (mess?) as for rods, but now the explicit determination of shortest paths is not so simple. (Observe that the space of rods is three-dimensional: the position of a rod is given by two coordinates of its green end and the angle with respect to a fixed direction. The space of triangles is twice as big: its dimension is six: two degrees of freedom for the position of an end, two spherical coordinates for the second end, and an angle for the third.)

What is so good about such spaces, what is the use for our distance? Here is one of the essential reasons for introducing these spaces.

The distance we defined is the distance chosen by Nature as recorded by the laws of Newtonian mechanics: if we forget about friction and give the rod a push, it will slide on the plane exactly along our shortest path. Similarly, a triangle in space, away from gravity, will also move along a path prescribed by our complicated definition. (This needs a little correction: in general, a free mechanical motion is truly shortest only for not very long time intervals. For example, a rod freely rotating around a fixed center follows the shortest path insofar as the rotation angle doesn't exceed 180 degrees.)

The same applies to every imaginable mechanical contraption, any system of rods, triangles and whatever joined at some points and left free to rotate around these points. Every free motion of such a thing in space will follow the shortest (to be precise, locally shortest) path in the corresponding multidimensional space. Thus knowledge of the geometry of such spaces is indispensable in practical mechanics and engineering (where one often has to modify the geometry in order to incorporate gravity and other forces).

The spaces of the above type are very diverse and their geometry may be quite complicated. Yet, one can establish and describe their properties in the unified language of distance. Engineers are concerned with specific instances of these spaces needed for concrete practical problems, e.g., designing a motion moving a system from one position to another. Mathematicians, on the contrary, are after properties common to all such spaces. Here is an instance of this: given two equal triangles in space, there always exists a shortest path between them. (Remember, “shortest” refers to the distance in the six-dimensional space of triangles defined in the same manner we introduced the distance in the space of rods in the plane.)

In mechanical terms, one can give one triangle an initial push, such that in the course of its motion it will take, at some moment, the exact position of another triangle.

This is also true in the presence of gravity: one can throw such a triangle at a wall, such that it hits the wall by its (say first) vertex and at the moment of collision will have a prescribed orientation in space. (It may be hard to do it if you try, but one can design a throwing machine doing it perfectly well, where the very possibility of such design is guaranteed by the general theorem.)

We conclude with a more complicated example, namely, we look at parallel parking of a car where the space where you have to squeeze in is barely sufficient. Even if the eventual position of the car is quite close to the original position one may need many goings back and forth in order to perfectly park.

The relevant space here is our old rod-space: the position of a car is determined two points, say two marks, one is at the front and the other at the back of the car. However, the relevant distance is quite different due to constraints on how a car can move. Unlike the sliding rod, one cannot move a car arbitrarily, e.g., sideways (unless the front and the back wheels can be turned 90 degrees). Yet, as we know from practice, and this confirmed by a mathematical theorem, one still can move a car to a given position, but it may take much longer than for the freely sliding rod.

It is more difficult, but yet possible, to park a car with a trailer, and an experienced drive can do this. Here, the corresponding space can be viewed as the space of two joined rods, and it has dimension four. Moreover, the parking problem admits a mathematical solution for any number of trailers. It is unlikely a driver can ever learn how to do this. Yet similar problems that appear in control theory of mechanical devices, e.g., in robotics, can be resolved with computer programs relying on geometry of the corresponding multidimensional spaces.

All of the above is supposed to give you a feeling of how geometric concepts are being created and applied. The geometric approach is not omnipotent; yet it often helps in solving difficult practical problems arising in mathematics, science and engineering.

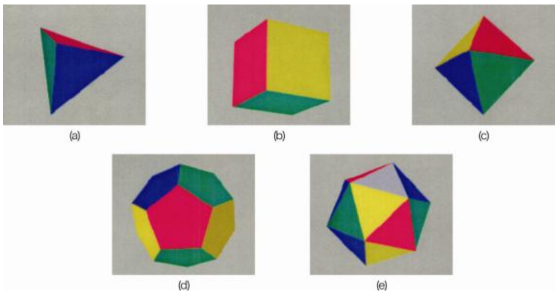


図 1　５つのプラトン立体
Fig. 1 Five Platonian solids

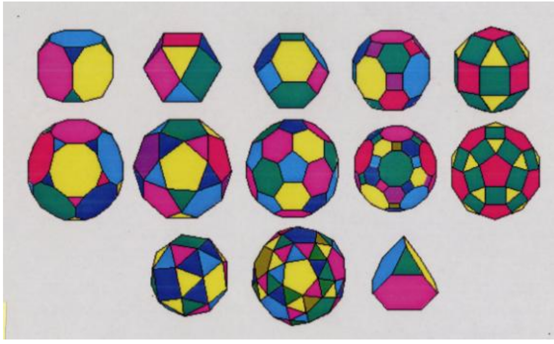

図 2　13のアルキメデス立体
Fig. 2 Thirteen Archimedean solids

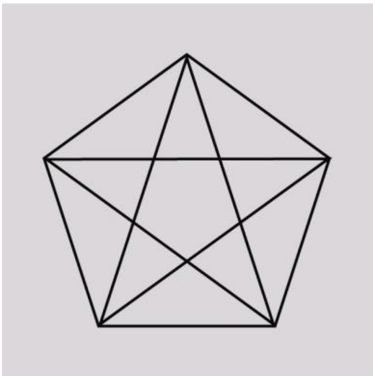

図 3　アルキメデスが見落とした完璧な立体
Fig. 3 Perfect body missed by Archimedes