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Elliott H. Lieb

My Journey Through Physics and Mathematics

Although I was born in Boston, USA in 1932, which was *Showa nana* (7), it was in New York City that I grew up and formed my view of the world. My family was middle class, but New York City offered a very good quality, free public school education.

I discovered that I liked to build things and to participate in Amateur or "Ham" Radio construction. What I was most proud of was learning the Morse code well enough to pass the examination for a license W2ZHS, to transmit and connect with other operators across the globe.

These early efforts in which my older cousin was influential was leading me, I thought, to a career in electrical engineering. When I was 17 years old, my family moved back to Boston and I was privileged with encouragement from the eminent physicist Viktor Weisskopf to go to the Massachusetts Institute of Technology, MIT, some people call it (Fig. 1).

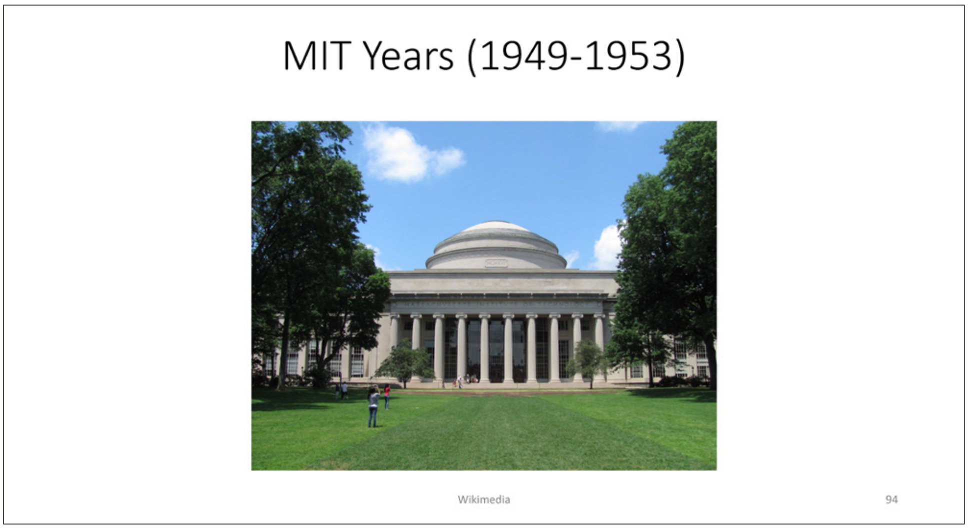

Fig. 1

Shortly after entering MIT in 1949, my outlook changed during my very first physics course. Matthew Sands, who co-authored the influential "Feynman Lectures," showed me the intellectual beauty of Newton's physics. At first, I had difficulty grasping the contents because my high school classes had not yet really prepared me for the deep understanding that a theorem such as Newton's equation holds.

The thing about Newton's equations, which took me a while to understand, is that it means exactly what it says: Force (F) = Mass (M) × Acceleration (A). In order to find (A), the acceleration, one must first find the magnitudes of the force and the mass. In different applications, these will be different, but the idea is the same. It required half the duration of the course plus Matthew Sands' patient help, but I finally understood the significance of the equation, and my career in science was off to a good start. Newton's contemporaries must have had similar difficulties. Thus, I gave up engineering and changed to a pure physics course for the rest of my undergraduate studies.

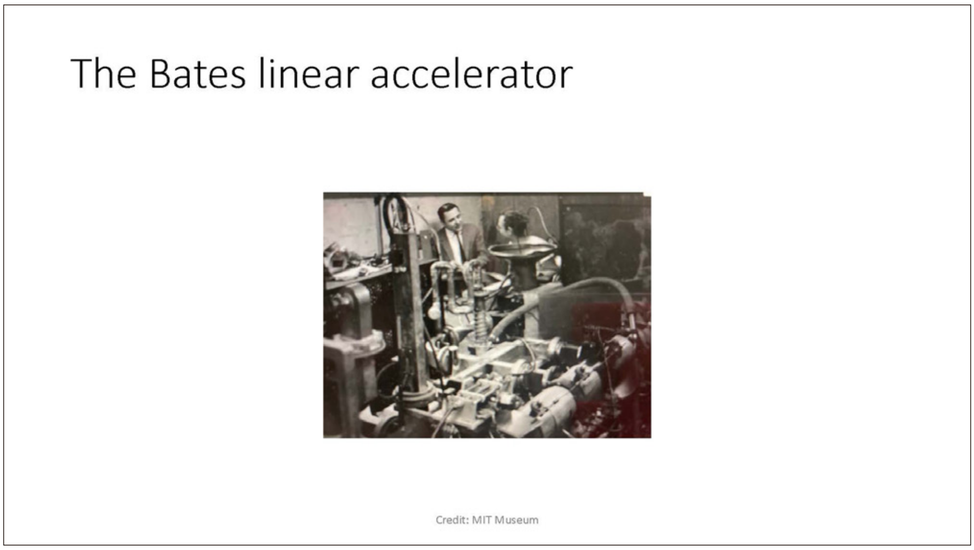

Fig. 2 Courtesy MIT Museum

As an undergraduate at MIT, I was lucky enough to get a part-time job in the group, working on the very first linear accelerator, which is a machine that is now very huge in application, but in those days was very small (Fig. 2). This 17 million electron volt machine, which was big then, but infinitesimal now, was built by two of the most encouraging people I ever met, Isaac Halpern and Peter Demos, and they had a big influence on my undergraduate perspective. Unlike the science of chemistry, physics was a subject not very well known to the general public in 1949. And my father thought that my sudden career decision into physics would lead me into poverty. It turned out otherwise, and I was lucky to ride the post-war wave of government funding of science with a decent standard of living. Mathematics played a role in my studies, but not a big one. I was most fortunate to learn advanced linear algebra at MIT from Isadore Singer of the Atiyah-Singer index theorem fame, and he became a very good friend.

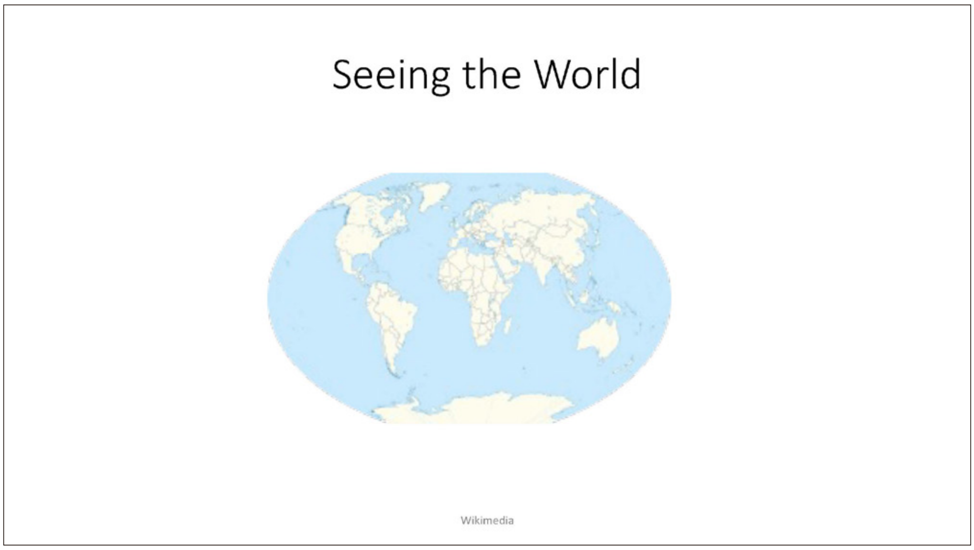

Fig. 3 Public Domain

After graduating from MIT, I wanted to get out of my cocoon and see something of the world (Fig. 3). Up to that point in time, I had traveled rather little apart from visits to some large American cities. Professor Weisskopf, with whom I also did my senior thesis on relativity theory, thought that the Department of Mathematical Physics at Birmingham University in England with

Professor Rudolf Peierls and lecturers Sam Edwards and Gerry Brown was one of the best places in Europe to do theoretical physics (Fig. 4).

Fig. 4 Courtesy University of Birmingham

And they spoke English, which was important for me since I had no exposure at that time to any other language. John Bell, who went on to invent extremely important inequalities in quantum information theory, was a fellow student. During those years, I did manage to fulfill my quest to visit most of Europe. After three years at Birmingham, with an uninspired Ph.D. thesis of little value, I graduated in 1956 with a Ph.D. degree. My next stop was Kyoto (Fig. 5).

Fig. 5 © Japan Guide

So, my first position after the Ph.D. was to go to Japan. Why Kyoto? My uncle, who owned an art bookstore in Boston, Massachusetts in the United States that specialized in Japanese art inspired in me a deep interest in *Ukiyo-e*. In Birmingham, I was lucky to share an office with a Japanese nuclear physicist named Shiro Yoshida, or Yoshida Shiro, I should say. I was asked to help him improve his English, which I did, and he reciprocated by teaching me some elementary Japanese without any *kanji* just by ear. So, I spoke Japanese as an illiterate.

Fig. 6

Nevertheless, the U.S. Fulbright Program gave me a one-year fellowship to Kyoto University's Yukawa Hall, or Yukawa Institute for Theoretical Physics, also known as Yukawa Kinenkan (Fig. 6). That year was formative for me, culturally and scientifically. Up to that point, I wondered if I would ever be able to contribute anything of value to science. In Kyoto, I did just that, or so I felt. After Kyoto, it took another four years to do so again. In the Yukawa Institute, I met a brilliant young Japanese physicist, Yamazaki Kazuo, with whom I formed a close collaborative association. Together, we worked on a challenging problem in physics, namely the Polaron model of an electron trapped inside a crystal (Fig. 7).

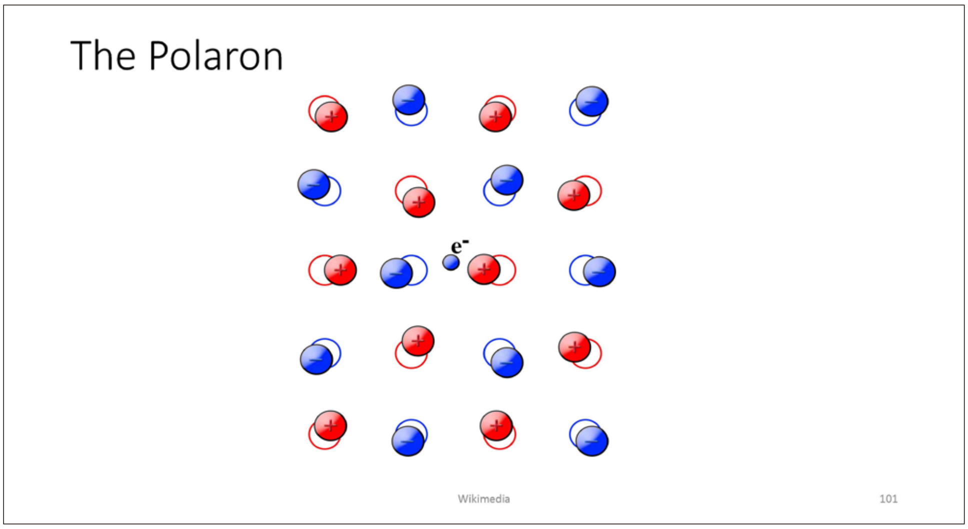

Fig. 7

It was a popular topic at the time, and we decided that we were going to go beyond intuitively motivated calculations and try to calculate the Polaron's lowest energy state in a mathematically rigorous way. We were able to prove that the Polaron energy was actually finite. In other words, that the lowest energy exists. Other physicists like Feynman took this for granted, despite the fact that similar looking models fail to have a finite energy, finite lowest energy. In this way, we opened a new chapter in our lives as well as in theory of the Polaron. Thanks to my stay in Kyoto, I was now convinced that I could actually do research.

Two years later, I met Richard Feynman himself at Cornell University, and he asked me about my interests. When I told him proudly what I had done in Kyoto with Yamazaki, his rather aggressive reply was that "Real physicists don't do that sort of thing!" In his opinion, I was wasting my time. As a young beginner, that negative encounter set my every sail ever more firmly toward mathematical physics and its importance to physics.

Fig. 8

Fig. 9

From Kyoto, I went for a year to the University of Illinois (Fig. 8), followed by two years at Cornell University (Fig. 9). Under the famous Nobel Prize winner, Professor Hans Bethe, who was the one who explained the nuclear reactions that makes the sun shine. Alas, those three years produced nothing and made me uncertain about my future as a mathematical physicist. But it did lead me to a problem that occupied me for the rest of my life, namely the quantum mechanics of a certain gas of atoms called a Bose gas, after the Indian physicist Bose, especially its coldest or lowest energy state (Fig. 10).

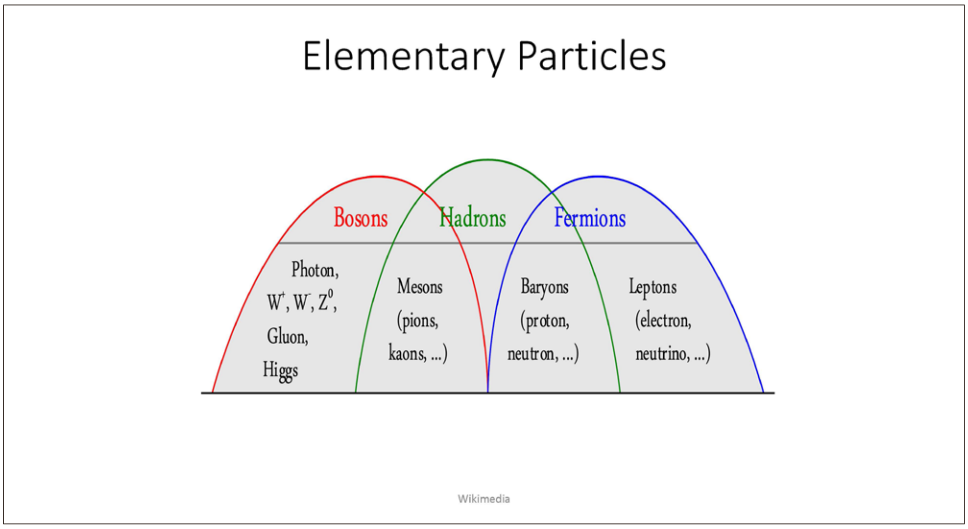

Fig. 10

Thus, I spent two years at the best institutions of its kind with one of the world's greatest physicists and came away in 1960 with nothing to show for it except for a problem to think about. It laid in my mind for 36 years until Jakob Yngvason and I solved this boson problem in 1996, which is many years later, and thereby started the current interest in Bose gases among mathematical physicists.

From Cornell, I moved to the IBM Computer Research Center in Yorktown Heights, New York, which had just opened that year, 1960. It was my first permanent position although I stayed with IBM for only three years. I was lucky to find two colleagues my age, Ted Schulze and Dan Mattis, and we three physicists felt that we wanted to prove some accepted theories by thinking mathematically. This interest of ours was then outside the realm of research in any other industrial research laboratory, and we were thankful to IBM for giving us the freedom to do so. In general, 1960 to 1970 was a wonderful decade for physics research around the world.

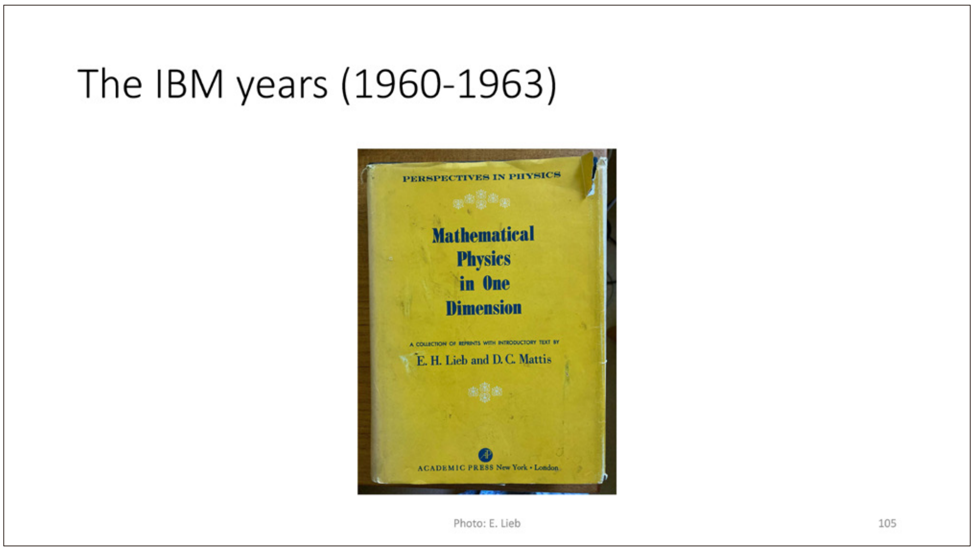

Fig. 11

Reprinted from Mathematical Physics in One Dimension, Elliott H. Lieb and Daniel C. Mattis, 1966

Several theorems of importance to physics came from this period (Fig. 11). One was the "Lieb-Mattis theorem," that magnetism never occurs in one dimension, that is to say, in a chain of atoms they will never become magnetic. There must be at least two dimensions. Most theorists at the time, including Heisenberg, the famous German physicist, and my old Birmingham Ph.D. advisor, Professor Peirels, and my Cornell mentor, Bethe, thought the exact opposite, that ferromagnetism always occurs in one dimension. And we had proved that it never occurs. And it took some effort to convince colleagues, that very important colleagues, Bethe and Peirels and Heisenberg, that we were correct, that it never occurs. And Peirels finally accepted our proof, this mathematical proof, one of the first mathematical proofs of general interest in quantum mechanics. And we developed several successive theorems about that. In the second of those three years at IBM, I was on leave in Sierra Leone (Fig. 12).

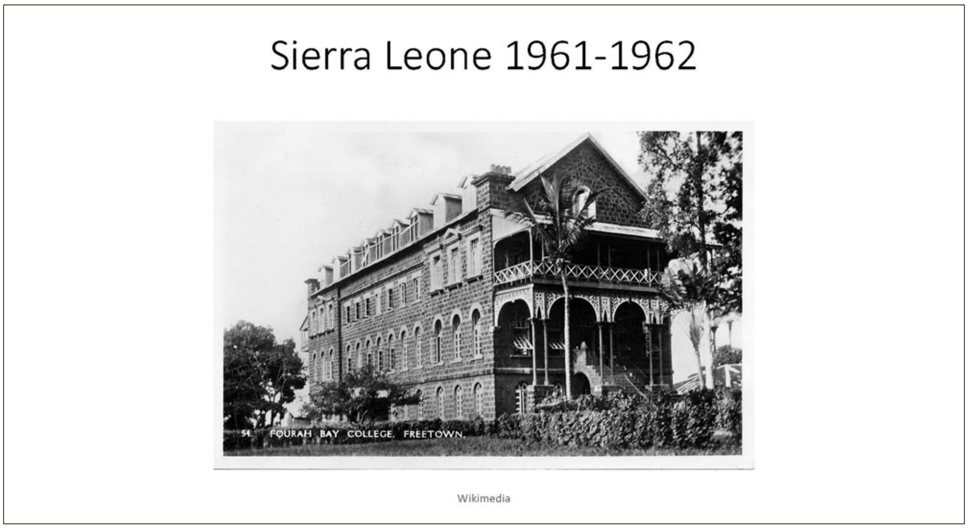

Fig. 12

Sierra Leone is in West Africa. I was in Sierra Leone for a year to teach applied mathematics at the university in Freetown. There were many adventures of a social political kind there, as well as a bout of malaria. And I also had time to think about science. Malaria, by the way, if you've never had it, is extremely unpleasant. It was there that I invented the model of Bose particles or 'bosons' in one-dimension, which I worked out later with Werner Liniger when I returned to IBM. This model is fundamental, is now fundamental for understanding the quantum mechanical many-body problem, and it has been confirmed experimentally, even though our model is only for a chain of atoms. After two years at Yeshiva University in New York, I moved to Boston again to take a professorship at Northeastern University. There, Professor F. Y. Wu and I wrote the most cited ever paper in the journal *Physical Review Letters* about the solution of the one-dimensional Hubbard model (Fig. 13). It still holds that citation record today. That is, it is the most cited ever paper in that particular journal.

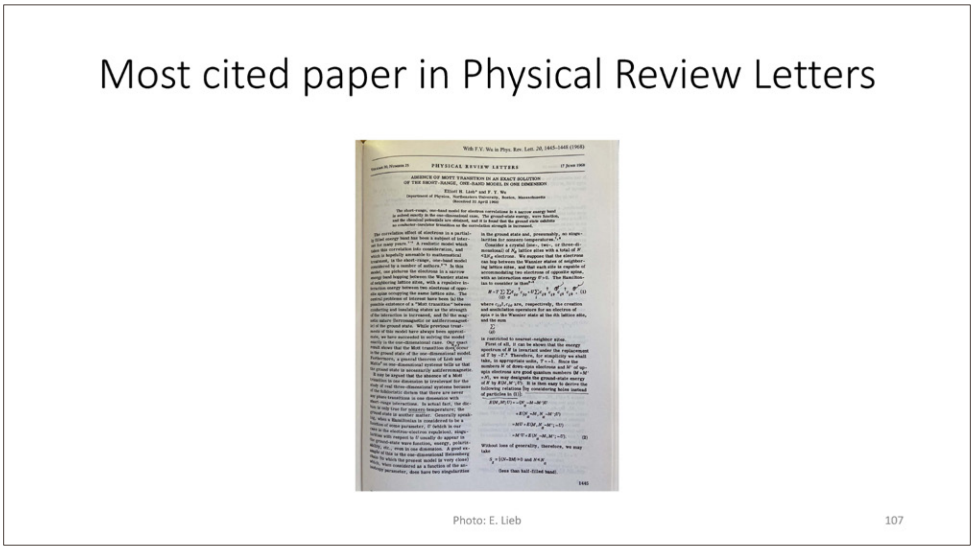

Fig. 13

I now turn my interest to something else at Northeastern University, namely to ice (Fig. 14). Ice is what you get when water freezes but it is not simple. What does ice have to do with mathematics? Well, one other important contributions here comes from Linus Pauling, who said that the entropy of ice can be computed by thinking about or computing the number of ways to arrange water molecules. Remember, water molecules are two hydrogens and one oxygen.

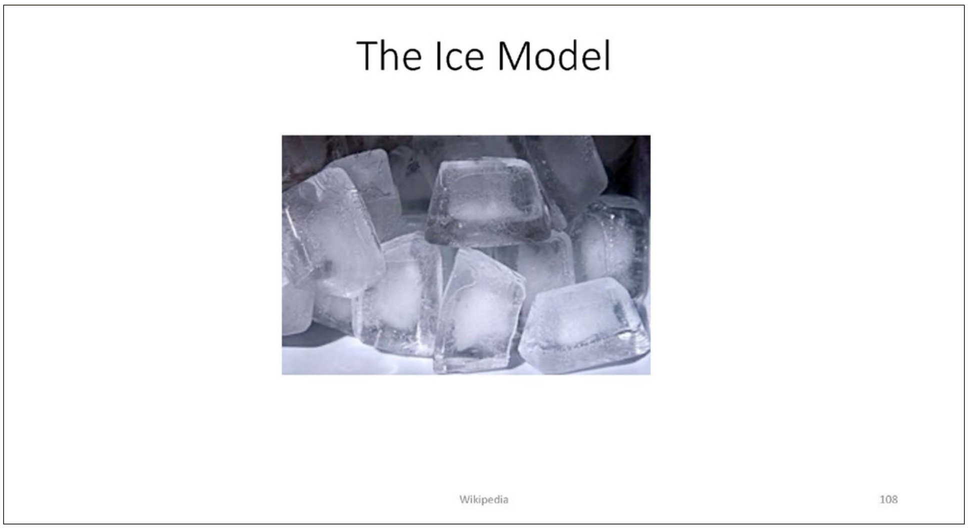

Fig. 14

Now, it was discovered experimentally in one of the most brilliant experiments ever done in physics that the entropy of ice does not go to zero at zero temperature. There is still entropy locked in, that just never goes away. This means that there is a significant randomness in the arrangement of the oxygen and hydrogen in ice. It's not just a regular arrangement but there's a lot of randomness in the way the water molecules are oriented toward each other.

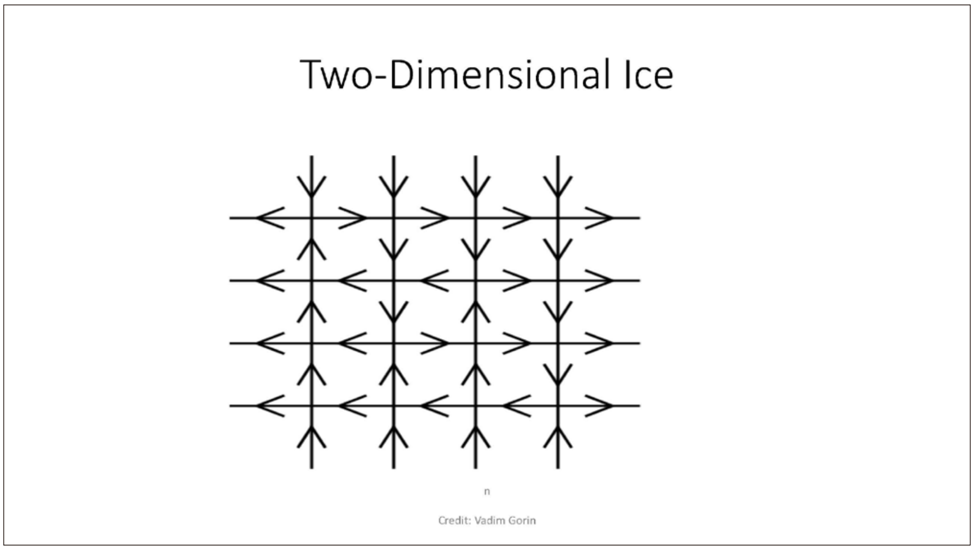

Fig. 15

And one way to describe this is to just think of a lattice model made up of arrows (Fig. 15). The points in the model here, the points are oxygen atoms and the arrows are the position of the hydrogen, which sits between the two oxygens and can be on one side or the other. And as I said, there is still some life left even at absolute zero. There's still some arrangements to be considered and counted, and that's what I set out to do. This model, by the way, was invented by Linus Pauling. The entropy to be calculated is the logarithm of the number of ways to make such an arrangement of hydrogen atoms given the oxygen atoms sitting on a regular lattice, as you see in this picture. The ice rule requires that two arrows are pointing into every vertex and two arrows are pointing out, away from every vertex. And the question is to figure out the number of ways you can put the arrows in this picture, in such a way that two come in and two go out at every vertex.

The number of ways turns out to be the number $\left(\frac{4}{3}\right)^{3N/2}$. And $\left(\frac{4}{3}\right)^{3/2}$ is 1.54, which is, that constant $\left(\frac{4}{3}\right)^{3/2}$ is now known, and is known after my name (Fig. 16).

This result gave rise to, gave birth to a whole subfield in combinatorics called the six vertex problem because if you have two arrows pointing in to a vertex and two arrows pointing out of a vertex, then the number of arrows configurations at each vertex, the number of them is six. There is six ways you can do this and you have to now count the ways of doing this at every vertex simultaneously. And that became a whole field in combinatorics and my contribution was to figure out the number of ways, but there are other things you can, other questions you can ask, and there are many open questions still in this area, even today.

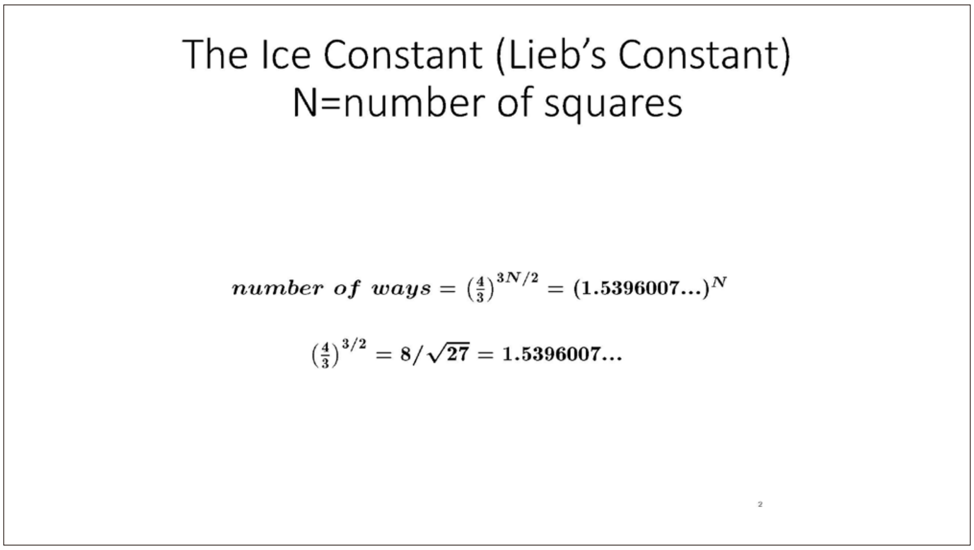

Fig. 16

The next few years was a high point in my work with Joel Lebowitz on the existence of the "thermodynamic limit for Coulomb forces." This theorem, together with the Freeman Dyson and Andrew Lenard proof of the energy lower bound for electrically charged particles, proved the 'stability of matter.'

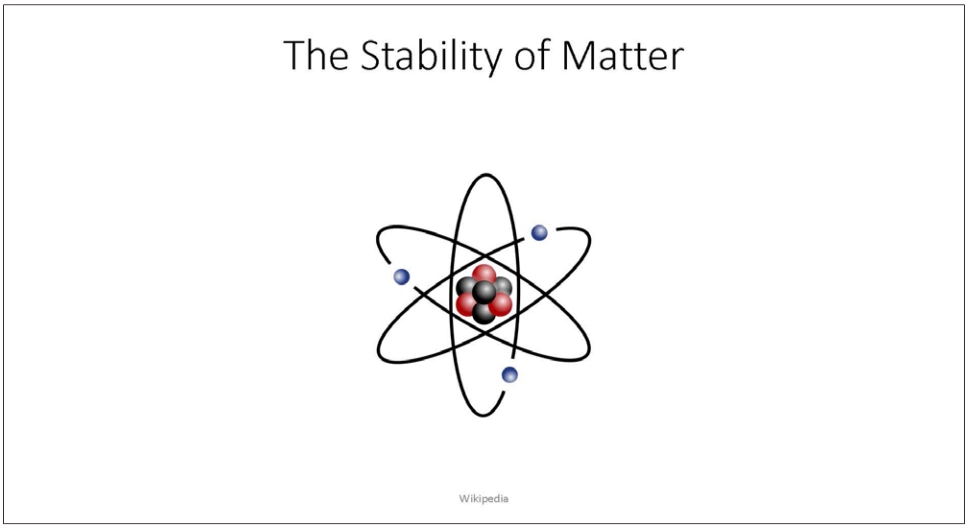

Fig. 17

Now, let me explain that word just a little bit. An atom, as you see in this picture (Fig. 17), has a nucleus and around the nucleus, electrons revolve. And there are as many electrons as there are the charge of the nucleus. Now you have to put all these atoms together and make a macroscopic amount of water, of anything, whatever the atom is made up of. And the question is, why doesn't this whole arrangement of nuclei and electrons, which looks like something a spider might have made, it looks so unstable, it looks like nothing is holding it together, why does this thing, which is made of essentially an infinite number of nuclei and electrons, why does this thing have the stability it has? You can knock on it and it doesn't break up, but it's made up of things that look like that picture on the board. And that question is a question that became known in physics only slowly. And, but we decided to tackle it. The picture is crude, but not wrong, the picture of the atom. And these atoms can attract each other, but only very slightly, and they retain their identities.

Unraveling this story took several decades of mathematical research and I contributed to some of this. This solution to this problem required contributions from several people. And one of the most important, as I mentioned, was Dyson and Lenard, and my colleague, there's also Walter Thirring.

In 1973, moving on, Mary-Beth Ruskai and I proved the Strong Subadditivity of Quantum Entropy. That is one of the cornerstones of quantum computation from the mathematical point of view, the strong subadditivity, the entropy of a quantum network. Now, this proof required a heavy dose of mathematical analysis, however, and thus began my period of pure functional analysis. Examples of my other work in this time period are another analytic set of inequalities: The Brascamp-Lieb inequalities, as they are now known, which are very, very much used in quantum information theory. The Brascamp-Lieb inequalities. Brasscamp was a young Dutch mathematician, a physicist, and we worked together.

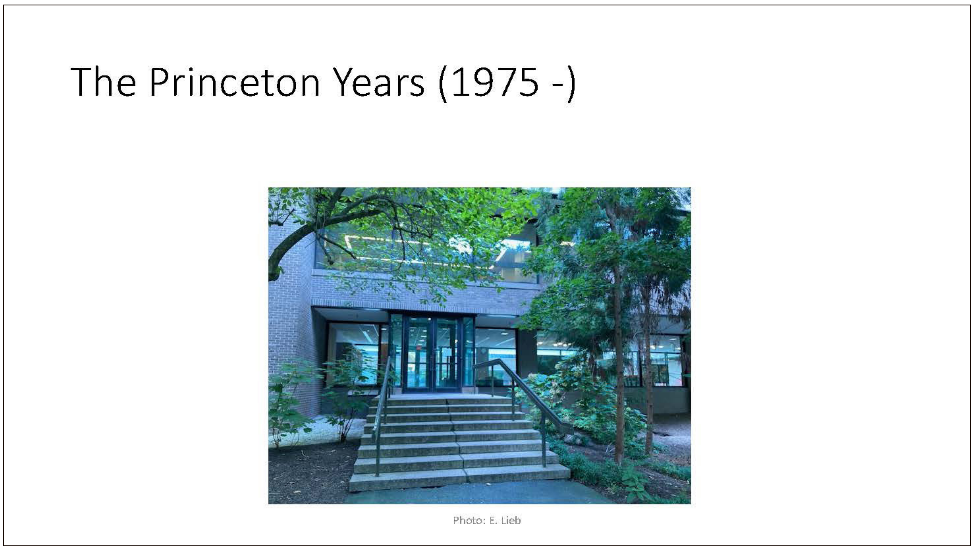

Fig. 18

In 1975, moving on now, I accepted an offer from Princeton University, jointly in its Mathematics and Physics Departments (Fig. 18). In the same year, Walter Thirring of the University of Vienna, one of the world's most famous mathematical physicists, and I struck up a friendship and set to work to find an alternative proof of this Dyson-Lenard proof of stability, which was rather complicated, actually. And we felt that a simpler one should be possible and not take so many pages and come up with a best, much better estimate for the stability itself. And we succeeded very much in doing this and we had to invent a very new line of mathematical inequalities which now go under both our names Lieb and Thirring.

Now let me mention some other things that occurred later. One of the more successful ones was something called the Lieb-Oxford bound. With [Steven] Oxford, we found the bound on which had not even been imagined that it existed, never mind finding it, for the exchange energy in solids. And this is the energy, I will not try to explain that, but it's the energy that holds a solid together, let's put it that way. Was [there] any kind of estimate on the limit to this? How big this could possibly be? And we found such an estimate, which was not expected.

Fig. 19

Now, we we're now in 1979, and I was fortunate to have a sabbatical year in Kyoto again with my wife, Christiane Fellbaum, who is sitting here (Fig. 19). And we had very, many exciting things happen to us but maybe one of the most important is the *densha*. We saw the last Kyoto *densha* come to a stop at Imadegawa. And this was quite an exciting event. There were a lot of people there and I remember very vividly this *densha* coming down the track and suddenly stopping. And that was it. That was the last *densha*.

Now, this bound, mathematical bound that I mentioned, it occurred in Kyoto, but with [Steven] Oxford at Princeton, we improved that to the current value. The second equally influential paper that I participated in also has a touch of Japan. This is the AKLT model of electrons spins. A is for Affleck, K is for Kennedy, Tom Kennedy, and the T, the T is for Tazaki, Tazaki Haruaki. Tazaki was a post-doctoral student of physics, researcher, I should say, with me in 1987. And he is one of the people who is here today. And this was one of the very first [Interacting] models in condensed matter physics that had an energy gap between the lowest energy state and the next available energy state. Very few substances have this property. Usually it's a continuum from the bottom upwards, but here there's a gap. And that gap is now very important in the electronics that you are very used to using.

And finally, I'll mention the one very last thing, which is my work with Jakob Yngvason on the meaning of thermodynamic entropy. Entropy is one of the oldest things in thermodynamics. It goes back to the beginnings of thermodynamics at the beginning of the 19th century. And what does it mean? Does it have a meaning? Apart from the fact that it's something you can measure, does it have a meaning independent of a picture designed by Boltzmann that its atoms and molecules bouncing around? That's what entropy is, now, it has a general meaning because it occurs in many different places nowadays in computer science, for example. And this we did, we found a meaning for entropy as the indicator of what transitions are possible. That's what entropy means. It's a way... Totally independently of any model you may make about what is an entropy physically, what does it mean? It means what is possible and what is not possible depending on whether the entropy you start with is less than the entropy you end up with. That's what it's meaning. It's a way of counting, saying that all possible states of matter have the property that you can reach from one state to another. But in most cases you can only go one way and that one way is determined by a simple function called the entropy. So this was a new approach or understanding of the subject.

In this talk, I have touched on several areas of mathematics and physics and the combination of the two. I have been fortunate to interact with many exceptional colleagues in several countries, and especially Kyoto, Japan, and to have received enough support and encouragement along the way to persist despite initial doubts. It is with humility that I thank the Inamori Foundation for awarding me a Kyoto Prize and for the opportunity to present these reflections on my life and work. Thank you.

You can watch the interview video after the commemorative lecture on the Kyoto Prize YouTube channel.
(https://www.youtube.com/watch?v=EvJdujHyqAU&list=PLmPcz49VTBBerReYZr-IE9kPl7SFvB3_F&index=6)